Eur. Phys. J. A **8**, 81–86 (2000)

THE EUROPEAN
PHYSICAL JOURNAL A

Statistical analysis of rare events—synthesis of the element 114

V.B. Zlokazov[a]

FLNP, JINR, 141980 Dubna, Moscow region, Russia

Received: 24 January 2000 / Revised version: 23 March 2000
Communicated by B. Povh

Abstract. Rare events pose a problem: is an observed chain of radioactive decays that of the background, or are they genetically linked? The paper suggests an approach for the problem solution, based on formalization of the background concept. This approach is an inevitable alternative to other methods, which require the *a priori* information about the linked decays, in a situation when such information is absent, but, instead, the background information is available, *e.g.,* from the calibration measurements. The method is illustrated by the analysis of data registered in the experiment on the synthesis of the element 114 as one of practically important examples of the analysis of rare events.

The logic and the apparatus of nuclear experiments getting ever more complicated, the situation arises when the result of such experiments is the observation of one single event, which admits a multiple interpretation, and first of all, as a random signal combination.

The direct use of statistical methods in this case is either impossible, or inappropriate: a combination with methods of probability theory is needed. Of course, the mathematical analysis in such situation loses the reliability and safety of the classical statistics; but it allows us to extract the optimum volume of information from the data of a small size which is possible in this case at all.

The observed event is in the best case a sequence of some subevents, or otherwise, signals, which cannot be identified even formally in the sense that they almost all are results of some radioactive decays, statistically independent of each other, and it is very difficult to decide from what decays they come—from those of interest or some others.

So both background signals and those of genetically linked decays of interest are random, statistically independent and formally undistinguishable. The only chance to separate them is given by differences of time characteristics of their combinations, or, speaking more generally, by the differences of probabilistic characteristics (means, variances, frequencies, etc.) of these combinations.

Bearing this in mind, one can build two approaches to tackle with the problem of signal identification:

– Formalize the concept of a background signal combination (BSC) and test whether the signal sequence analyzed does fit in this concept or not.
– Formalize the concept of a linked decay signal combination (LDSC) and test whether the signal sequence analyzed does fit in this concept or not.

[a] e-mail: `zlokazov@main1.jinr.dubna.su`

The authors of [1] (Dr. K.H. Schmidt and his colleagues) have preferred the latter approach (correlation analysis in their terms). Their method is widely used now, but its success strongly depends on the volume and quality of the *a priori* information about the qualitative structure of the event (list of classes in terms of [1]) and the half-lives of its constituents.

In cases of extremely indefinite and poor experimental outcomes we do not know the structure of the decay chain *a priori*; neither the reliable information about the half-lives of members of this chain is available. In this situation the first approach is more attractive: see whether the signal group analyzed corresponds statistically to the pattern of the BSC or not; if not then the next analysis comes trying to find the informative pattern for the event interpretation. The most important advantage of this approach is the fact that for building the BSC pattern we can use the objective sources of information—data of the background calibration measurement, which, in addition, are not affected by poor statistics. The first approach is not competitive against the second one; rather, it is a necessity to which one resorts when there is a complete absence of reliable information about the characteristics of the physical process.

Below some mathematical tools needed for further analysis are described.

1 Functions of probability distribution for the radioactive decay

The classical function of probability distribution for an event (the radioactive decay) at a time moment t is: $P(t) = 1 - \exp(-lt)$, where $l = \ln(2)/T$, and $T =$ the half-life of the nucleus. The density of this probability is

$f(t) = l \cdot \exp(-lt)$. With the help of $P(t)$ and $f(t)$ all the other distributions can be obtained.

So the function of the probability distribution for the daugther nucleus is

$$P_{12}(t) = \int_0^t f_1(\tau) P_2(t-\tau)\,\mathrm{d}\tau$$

and substituting the concrete expressions in $P_2(t)$ and $f_1(t)$ we get

$$P_{12}(t) = \begin{cases} 1 - \exp(-l_1 t) - l_1/(l_1 - l_2) & \\ \quad \cdot(\exp(-l_2 t) - \exp(-l_1 t)), & \text{if } l_2 \neq l_1, \\ 1 - \exp(-l_1 t) - l_1 t \cdot \exp(-l_2 t), & \text{if } l_2 = l_1. \end{cases}$$

Similarly, the same functions are built for the successors of the consequent decay of the original nucleus:

$$P_{123}(t) = \int_0^t f_1(\tau) P_{23}(t-\tau)\,\mathrm{d}\tau, \tag{1}$$

$$P_{1234}(t) = \int_0^t f_{12}(\tau) P_{34}(t-\tau)\,\mathrm{d}\tau, \tag{2}$$

$$P_{12345}(t) = \int_0^t f_{123}(\tau) P_{45}(t-\tau)\,\mathrm{d}\tau, \tag{3}$$

where $1, 2, 3, 4, 5$ denote mother, daugther, grand daugther etc., respectively. The computational procedure can use (1), (2), (3) directly in general form, since the concrete formulae are too bulky due to many combinations of coinciding and not coinciding half-lives.

2 Function of probability distribution for a quadratic form

Let the following quadratic form

$$S_n = \sum_{i=1}^{n} x_i^2 \tag{4}$$

be given, where normally distributed random quantities x_i have zero expectation, unit variance, and the neighbouring pairs x_i, x_{i+1} are correlated with the correlation coefficient -0.5. What is the function of probability distribution of S_n? One of the widely spread errors in the practice of data analysis is that this function is supposed to have the χ_n^2 distribution. In fact, it has not.

From the mathematical statistics [2] it is known that the quadratic form $Q_{ij} x_i x_j$, $i, j = 1, ..., n$, where Q is the inverse normal covariance matrix of x_i, has the χ_n^2 distribution, irrespective of whether the x_i are correlated or not; therefore, if we had taken the inverse matrix of

$$c_{ij} = \begin{pmatrix} 1 & -0.5 & 0 & \cdots & 0 \\ -0.5 & 1 & -0.5 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & -0.5 \\ 0 & 0 & \cdots & -0.5 & 1 \end{pmatrix} \tag{5}$$

and built a quadratic form

$$\hat{S}_n = \sum_{i,j=1}^{n} c_{ij}^{-1} x_i x_j,$$

it would have the χ_n^2 distribution. But the inversion of a matrix like (5) is rather complicated; besides, the probability of large deviations of a χ^2-distributed random quantity is substantially larger than that of S_n (*e.g.*, for $n = 4$ about 0.07 and 0.045, respectively) and the decision making procedure based on the use of $\hat{S}_n$ loses part of its efficiency specifically in this case.

Thus, preferable is a method based on the direct use of S_n. The probability distribution function and its density for (4) can be easily calculated numerically. One can show, that the expectation of (4) is equal to n, and the variance to $3n - 1$. Below a table of values of $P(t)$ for $n = 4$ is given.

P	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
S_4	.6	.9	1.1	1.4	1.6	1.9	2.2	2.5	2.8	3.1
P	.55	.60	.65	.70	.75	.80	.85	.90	.95	
S_4	3.4	3.8	4.2	4.7	5.3	6.0	6.9	8.2	10.3	

Numbers in each upper row are probabilities P with the step 0.05, and in the lower one the corresponding values of S_4.

With the help of such table (certainly, more detailed) we can construct the 67% confidence interval of S_4 as $n \pm \sigma$; corrected for the asymmetry it is: (1.8–10.8).

3 The formalism of stochastic Poisson time processes

These are the time functions $K(t_1, t_2)$—number of random events, occurred during a time interval (t_1, t_2) with a probability $Q_k(t_1, t_2)$ and having the following properties:

1. stationarity: $Q_k(t_1, t_2) = Q_k(t_2 - t_1)$ for arbitrary t_1, t_2;
2. $Q_k(t_1, t_2)$ independence of the event prehistory: $Q_k(t_1, t_2|C) = Q_k(t_1, t_2)$, where C means events which happened before t_1;
3. rareness of events: $Q_{k>1}(\delta t) = o(\delta t)$.

These properties allow us to write simply $K(t)$ and $Q_k(t)$ bearing in mind that t means the duration of the time interval considered.

The Poisson processes play an important role in analytical modelling of the stochastic time event background in scientific and technical applications because random events very often satisfy the above-numbered requirements. They disable the need to use the computer simulation to get the estimates of the random background characteristics.

The function of probability distribution of $K(t)$ is

$$Q_k(t) = \frac{(lt)^k}{k!} \exp(-lt), \tag{6}$$

where l = parameter of the Poisson distribution, t = time, and Q_k = probability that during a time interval $(0,t)$ k events will be registered.

The quantity lt is the expectation and at the same time the variance of $K(t)$ at a moment t.

Another independent characteristic of the random event sequence is T—time between the two subsequent events. As known [2], it is subject to the exponential distribution

$$P(T < t) = 1 - \exp(-lt), \tag{7}$$

where l is the same parameter as in (6). The expectation T_m and the variance T_v are

$$T_m = \frac{1}{l}, \qquad T_v = \frac{1}{l^2}. \tag{8}$$

The estimates of l, obtained from (8) and from (6) are almost uncorrelated, and, therefore, lt and T can serve as independent statistical characteristics of the data analyzed.

On the basis of (7), we can determine $T_{\max}$—the average maximum T, and $T_{\min}$—the average minimal T. Making use of the formula [3] for the expectation of the maximum of n random quantities x_i, $i = 1, 2, \ldots, n$, subjected to a distribution $F(x)$, $x \in (a,b)$:

$$\hat{E}\mathrm{Max}(x_i) = \int_a^b nxF(x)^{n-1}\frac{\mathrm{d}F}{\mathrm{d}x}\mathrm{d}x,$$

and the corresponding formula for the minimum, we get

$$T_{\max} = \int_0^\infty tnl\big(1 - \exp(-lt)\big)^{n-1}\exp(-lt)\,\mathrm{d}t,$$
$$T_{\min} = \int_0^\infty tnl\exp(-lnt)\,\mathrm{d}t, \tag{9}$$

where n = the number of items in the event sequence.

The formal analysis of both BSC and LDSC data does not show any qualitative difference between them; both satisfy the distributions (6) and (7). But with different parameters l! And this is the only chance to distinguish a BSC data from a LDSC one:

> *A BSC is distinguishable from a LDSC if the confidence intervals of their parameters l do not overlap.*

4 Application to the element 114

As example, let us consider the analysis of data obtained in the experiment on the synthesis of superheavy nuclei in the ^{48}Ca + ^{244}Pu reaction [4], in particular, of the 114th element. The data of interest (in a group of others, which are uninteresting) is a chain of registered signals, starting with the implantation of the recoil nucleus, followed by 3 alpha-decays, and ending with the spontaneous fission; all the signals were observed in the same strip of the detector. We can summarize these data in the following way.

The registered locations of the decaying nucleus in the position-sensitive detector are as follows [4]:
xevr = 16.5 mm—position of the implantation signal.
xa1 = 15.6 mm—position of the alpha-1 signal.
xa2 = 16.5 mm—position of the alpha-2 signal.
xa3 = 17.0 mm—position of the alpha-3 signal.
xsf = 17.1 mm—position of the spontaneous fission signal.
The span of these positions is about 1.6 mm.
Then we have "resolutions"—FWHMs of the distributions of signal differences, which, on assumption of the normal distribution of these differences, can be transformed into the usual sigmas:
EVR-alpha: resolution = 1.4 mm; $sig_{evr\text{-}\alpha} = 0.59$ mm;
alpha-alpha: resolution = 1.0 mm; $sig_{\alpha\text{-}\alpha} = 0.42$ mm;
EVR-SF: resolution = 1.2 mm; $sig_{evr\text{-}sf} = 0.51$ mm;
The following times were registered between the spontaneous-fission signal and the nearest foregoing implantation signal appearance:
t of implantation signal = 0;
t of alpha-1 signal = 0.5 min;
t of alpha-2 signal = 15.9 min;
t of alpha-3 signal = 17.5 min;
t of spontaneous fission signal = 34.0 min.
To complete this dataset, let us adduce the results of calibration measurement of chance signals of recoil implantation and alpha-particles with energy 8.5–10 in a detector strip for a position-correlation window 1.6 mm:
implantation = 1.3 per hour, alpha-particle = 1 per hour.

Now we start the analysis of all these data. We shall go over a set of possible interpretations of this data proposed by the physicists and consider the following problems: within the framework of these interpretations estimate the formal probabilities of the observed signal configuration and some of its statistical characteristics.

4.1 Interpretation 1: "events are random, and the spontaneous fission has no relation to the reaction $^{48}\mathrm{Ca} +^{244}\mathrm{Pu}$"

The above-mentioned time events represent a typical time process of the Poisson type (6) and we can use here its technique. Let us derive the probability of the events: one imitator of the implantation signal 34 minutes before the spontaneous fission, and 3 imitators of alpha-particles between them.

We estimate l in (6) for implantation and alpha-particle imitators on the basis of calibration data as follows:

$$l_i \cdot 60 = 1.3; \qquad l_\alpha \cdot 60 = 1.$$

Solving these equations, we get $l_i = \frac{1.3}{60}$; $l_\alpha = \frac{1}{60}$.

Substituting in (6), we get the probability of 3 alpha-particles

$$Q_3(34) = \frac{(34/60)^3}{3!}\exp\left(-\frac{34}{60}\right),$$

and the probability of one implantation 34 minutes before the spontaneous fission:

$$P_\tau(34) = 34 \cdot \frac{1.3}{60} \exp\left(-\left(34 \cdot \frac{1.3}{60}\right)\right).$$

Thus, we have the probability P_s for this data interpretation

$$P_s = Q_3 \cdot P_\tau \sim 0.00607.$$

This probability does not give yet a notion about the likelihood of the considered signal interpretation—to make a statistically correct decision it is necessary to compare it with the probabilities of other random signal combinations. We have

Imp.	Alpha	Probability
1	0	0.20010
1	1	0.11339
1	2	0.03213
1	4	0.00086
2	1	0.04176
2	0	0.07370

Here Imp = number of signals for the implantation, Alpha = for the number of alpha-particles, Probability = probability of such combination. It is seen that the largest is the probability to observe the combinations $1+0$ and $1+1$, but the probability of the combination $1+3$ is really small as compared with them. In other words, the data obviously contradicts the hypothesis about the background character of signal emergence.

The information about the time characteristics of the sequence analyzed complies with this conclusion. Calculating T_m (8) for both the background and the chain of interest, we get

background: $T_m = 26.09\,\text{min}$,
tested sequence: $T_m = 8.5 \pm 3.7\,\text{min}$.

The first T_m is obtained from the calibration data and its accuracy can be made very high. One can see that both T_m differ very strongly from one another.

In addition to this, let us calculate the average $T_{\max}$ and $T_{\min}$ for both the data on the basis of (9): ($n = 4$)

$$\begin{aligned}\text{background: } T_{\max} &= \frac{25}{12} \cdot T_m = 54.4,\\ T_{\min} &= \frac{1}{4} T_m = 6.5,\\ \text{tested sequence: } T_{\max} &= \frac{25}{12} \cdot T_m = 17.7,\\ T_{\min} &= \frac{1}{4} T_m = 2.1.\end{aligned}$$

To confirm the above conclusion, we can make use of the spatial difference between a BSC and a LDSC: the successors of a decay chain remain at the same place, whereas random events are scattered on the surface detector randomly.

Let a hypothesis be tested: all the signals arise as a result of a decay of a parent nucleus, which is located at a fixed position in the detector strip.

The statistical test of this hypothesis can be carried out by two methods.

Method 1. Let us construct an expression

$$S = \left(\frac{xevr - xa1}{sig_{evr\text{-}\alpha}}\right)^2 + \left(\frac{xa1 - xa2}{sig_{\alpha\text{-}\alpha}}\right)^2 + \left(\frac{xa2 - xa3}{sig_{\alpha\text{-}\alpha}}\right)^2 + \left(\frac{xevr - xsf}{sig_{evr\text{-}sf}}\right)^2. \quad (10)$$

Substituting the corresponding values of variables into (10), we get: $S_4 = 9.56$. One sees at once that it is covered by the 67% confidence interval of the quantity S_4: (1.8–10.8).

Method 2. The analysis of the differences is less efficient than the analysis of their constituents, since the variance of the former is always greater than that of the latter. Besides, these variances are obtained from the calibration reactions, and can differ from the true variances of difference signals for the reaction considered.

Therefore, for a greater reliability we can use the classical approach of the statistics: analysis of the constituents of these differences. Let us find the sample mean and the variance for the signals of the nucleus position. We have for the mean

$$\text{pos} = \frac{(16.5 + 15.6 + 16.5 + 17.0 + 17.1)}{5} = 16.54.$$

For the sample variance we apply the usual formula:

$$\text{var} = \sum_{i=1}^{n} \frac{(x_i - \text{pos})^2}{n-1}.$$

After the necessary calculations we obtain var = 0.35, whence we find the sigma: $\sigma = 0.59$.

Let us consider the expression

$$Q = \sum_{i=1}^{5} \left(\frac{x_i}{\sigma}\right)^2, \quad (11)$$

where x_i = difference between i-th signal and pos.

If our hypothesis holds then under very common assumptions each quotient and their sum will have asymptotically the Student's and χ^2_3 distributions, respectively. Substituting our data in Q, we get $Q = 4.0$.

The expectation of this quantity is $\hat{E}Q = m = 3$, the variance $\hat{V}Q = 6$ and $\sigma = 2.45$. The 67% confidence interval calculated as $m \pm \sigma$ and corrected for the asymmetry is equal to (1.30, 7.50); our Q gets into it. Thus, both

methods find out that the data does not contradict the hypothesis that events are a LDSC.
Remark. The variance of the difference is twice greater than the variance of its constituents; since the sigmas of the differences are smaller than 0.59, it points out that real difference variances are larger than the given above. So, the real value of (10) is even smaller than 9.56.

4.2 Interpretation 2: "data is the result of the decay of element-114 recoil"

The main conclusion being made: "the signals most probably are not a BSC", we can further try to test several hypotheses about the possible physical meaning of this sequence. We start with the above one.

If this interpretation is valid the data is a sequence of the events: implantation of the recoil, 3 consequent alpha-decays and finally the spontaneous fission, and the quantitative analysis should consist in testing the correspondence of the observed energies and half-lives of the alpha-particles to calculations, given, *e.g.*, in [5].

We do not know the half-lives of the nuclei produced in the decay, but we have *a priori* estimates of the intervals containing these half-lives, and we can set the problem as follows: determine the maximum and minimum probability of the decay of grand daugther P_{1234} (2) within the time range from the signal of recoil implantation to the signal of the spontaneous fission over the direct product of confidence intervals for the half-lives.

The calculation of the maximum and minimum of (2) over the region

$$0.05 \le T_1 \le 0.5, \qquad 30 \le T_2 \le 300,$$
$$2 \le T_3 \le 20, \qquad 7 \le T_4 \le 27,$$

in the time interval (0, 34 min) with account of the registration efficiency, equal to 0.87, gave the following results:

$$P_{\min} = 0.0083, \qquad P_{\max} = 0.3364.$$

4.3 Interpretation 3: "the chain is decay of the element 112, and one alpha-particle is imitator"

Assuming that the recoil nucleus is the element 112—by $(\alpha, 3\mathrm{n})$-evaporation channel—and the first alpha-particle is imitator, the problem is: determine the maximum and minimum probability of the decay of the grand daughter P in the time interval from implantation signal to that of the spontaneous fission on the direct product of confidence intervals for the half-lives.

The maximum and minimum of (1) over the region

$$30 \le T_2 \le 300, \qquad 2 \le T_3 \le 20, \qquad 7 \le T_4 \le 27,$$

in the time interval (0, 34 min) gave the following results:

$$\hat{P}_{\min} = 0.0293, \qquad \hat{P}_{\max} = 0.4934.$$

Multiplying these probability by the probability of the imitation of one alpha-particle in the time interval (0, 34 min) $Q_1(34) = \frac{34}{60}\exp(-\frac{34}{60})$, and correcting them for the registration efficiency, we finally get

$$P_{\min} = 0.0032, \qquad P_{\max} = 0.1273.$$

4.4 Interpretation 4: "the chain is the decay of a product of the transfer reaction between the nuclei of the projectile and the target, and alpha-particles (all or part) are imitators"

This case is similar to the previous one, but the probabilities of the event configurations and the genetic connection between them will be smaller, and the more alpha-particles are suppposed to be imitators, the smaller. For instance, let us consider the case: 1 alpha-particle is true, the other 2 are imitators. Suppose that the half-life of the mother-nucleus is contained in the interval (1–100 min). Omitting the details of the calculations (they are similar to the above ones), we get

$$P_{\min} = 0.0089, \qquad P_{\max} = 0.1521.$$

The analysis of the results. We have

Interpretation number	Max. probability	Min. probability
2	0.3364	0.0083
3	0.1273	0.0032
4	0.1521	0.0083

One can see that the comparison of the formal probabilities to observe the signal configuration given does not contradict the preference for the interpretation 2 made by the authors of [4]. And this non-contradiction substantially increases if we attach the physical probabilities, by which the formal probabilities should be multiplied:

1. The cross-section of the channel $(\alpha, 3\mathrm{n})$ for interpretation 3 is several orders smaller than that of the channel (3n) for interpretation 2;
2. The probabilities to observe the energies of Alpha-particles given in [4] and possible half-lives for interpretation 4 are very small as compared with the probabilities for interpretation 2.

Unfortunately, the numerical evaluation of the physical probabilities is impossible, since the calculations like [5] do not contain confidence intervals for the possible energies of the alpha-decay and half-lives.

Still, we can make here the following

5 Conclusion

The performed analysis is an illustration of rare event treatment, which gives a rather reliable means to distinguish a background decay sequence from that of genetically linked decays under lack of information about the half-lives of constituents of the decaying chain when the method of [1] is not applicable.

This work has been performed at the request of Prof. Yu.Ts. Oganessian. The author thanks him and Drs. V.K. Utyonkov and A.G. Demin for the helpful discussions of the physical aspects of the problem.

References

1. K.H. Schmidt, C.C. Sahm, K. Pielenz, H.G. Clerc, Z. Phys. A **316**, 19 (1984).
2. S.S. Wilks, *Mathematical Statistics* (J. Wiley, London, New York, 1962).
3. S.V. Nagaev, *Mathematical Statistics* (Novosibirsk, NGU, 1973).
4. Yu.Ts. Oganessian, V.K. Utyonkov *et al.*, *The synthesis of superheavy nuclei in the* $^{48}\mathrm{Ca} + {}^{244}\mathrm{Pu}$ *reaction*, JINR, Preprint (E7-99-53, Dubna, 1999).
5. R. Smolanczuk, Phys. Rev. C **56**, 812 (1997).